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Permutation group and interacting subsystems of particles

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Abstract. A simple procedure is outlined for adapting the basis spanning the irreducible representations of the permutation group S_N to those spanning the product representations of the subgroup $S_{N_1} \otimes S_{N_2}$ where $N_1 + N_2 = N$. An algorithm based on the procedure is also discussed.

1. Introduction

Recent studies of multishell configurations of electrons in atoms (Harter and Patterson 1977) and interacting subsystems of electrons in molecules (Kaplan 1975, Sarma and Dinesha 1979) have led to the consideration of a basis spanning the irreducible representations (irreps) of the unitary group $U(n_1+n_2)$ symmetry adapted to the subgroups $U(n_1) \otimes U(n_2)$. The unitary transformation which relates the basis of $U(n)$ adapted to the subgroup chain $U(n) \supset U(n-1) \supset \ldots \supset U(1)$ to the one adapted to $U(n) \supset U(n_1) \otimes U(n_2)$ is relatively simple for the irreps $\langle 2^{\lambda_2}, 1^{\lambda_1-\lambda_2}\rangle$ $(\lambda_1+\lambda_2 = N, \lambda_1 - \lambda_2 = 2S)$ of interest in many-electron studies. For more general spin systems such as the nuclear spins used in hyperfine interaction studies (Harter and Patterson 1979), the problem tends to become quite complicated. The first step in the subgroup adaptation required in such cases is the efficient generation of a canonical basis for an arbitrary irrep $\langle \lambda \rangle \equiv \langle \lambda_1, \lambda_2, \ldots, \lambda_n \rangle$ $(\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n \geqslant 0)$ of $U(n)$. An algorithm has recently been developed by Sarma and Rettrup (1980a,b) for generating these basis states by computer. The second step in generating the non-canonical basis is to obtain the linear transformation relating these basis states of $U(n)$ to those spanning $\langle \mu \rangle \otimes \langle \nu \rangle$ of $U(n_1) \otimes U(n_2)$. Since this problem is quite complicated, it will be tackled in stages. As a first step we attempt, in the present note, the adaptation of the Young orthogonal basis of an irrep $[\lambda]$ of the permutation group S_N on N particles to the basis spanning the irreps of the subgroup $S_{N_1} \otimes S_{N_2}$ $(N_1 + N_2 = N)$. An algorithm has been obtained for determining the subduction coefficients for the restriction $[\lambda] \downarrow [\mu] \otimes [\nu]$ of $S_N \supset S_{N_1} \otimes S_{N_2}$. Introducing a carrier space for the representations of $U(n)$ it should be possible to extend the present considerations to the subgroup restriction of the unitary group. This aspect to the problem has, however, not been dealt with here.

A recursive scheme has been outlined in § 2 for the symmetry adaptation of the Young orthogonal basis for the irreps of S_N to the product basis of the subgroup $S_{N_1} \otimes S_{N_2}$. The procedure has been illustrated in § 3 using a number of examples. A brief discussion of the procedure has been presented in § 4.

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2. Reduction of outer product representations of $S_{N_1}\otimes S_{N_2}\subset S_N$

Let S_N be the permutation group on N identical particles and consider the subgroup $S_{N_1}\otimes S_{N_2}$ where $N_1+N_2=N$. Let $[\lambda]$ be an f_N^λ-dimensional representation of S_N and $[\mu]$, $[\nu]$ those of S_{N_1}, S_{N_2} respectively. This $f_{N_1}^\mu f_{N_2}^\nu$ dimensional irrep $[\mu]\otimes[\nu]$ of $S_{N_1}\otimes S_{N_2}$ yields on induction a representation of S_N which is decomposable into the irreps of S_N as

$$[\mu]\otimes[\nu]=\sum_{[\lambda]} a_{\mu\nu}^{\lambda}[\lambda] \tag{1}$$

where $a_{\mu\nu}^{\lambda}\geqslant 0$ is the multiplicity of occurrence of $[\lambda]$ in $[\mu]\otimes[\nu]$. Using the rules for reduction of outer products of S_N (Hamermesh 1962 p 250) it is straightforward to determine the irreps occurring on the right of equation (1). As an illustration consider the product $[2, 1]\otimes[2, 1]$ of $S_3\otimes S_3\subset S_6$ which can be diagrammatically realised as

$$\begin{array}{ll}\cdot&\cdot\end{array}\ \otimes\ \begin{array}{ll}a&a\\ b&\end{array}\ \rightarrow\left(\begin{array}{llll}\cdot&\cdot&a&a\\ \cdot&b&&\end{array}+\begin{array}{llll}\cdot&\cdot&a&a\\ \cdot&&&\\ b&&&\end{array}+\begin{array}{lll}\cdot&\cdot&a\\ \cdot&a&b\end{array}+\begin{array}{lll}\cdot&\cdot&a\\ \cdot&a&\\ b&&\end{array}\right.$$
$$\left.+\begin{array}{lll}\cdot&\cdot&a\\ \cdot&b&\\ a&&\end{array}+\begin{array}{lll}\cdot&\cdot&a\\ \cdot&&\\ a&&\\ b&&\end{array}+\begin{array}{ll}\cdot&\cdot\\ \cdot&a\\ a&b\end{array}+\begin{array}{ll}\cdot&\cdot\\ \cdot&a\\ a&\\ b&\end{array}\right)$$

where the notation and the rules used are those given by Hamermesh. Thus for this example equation (1) reduces to

$$[2, 1]\otimes[2, 1]=[4, 2]+[4, 1^2]+[3^2]+2[3, 2, 1]+[3, 1^2]+[2^3]+[2^2, 1^2].$$

A dimensionality check on equation (1) is provided by the formula (Robinson 1961 p 55)

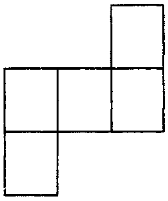
$$\frac{N!}{N_1!N_2!}f_{N_1}^{\mu}f_{N_2}^{\nu}=\sum_{\lambda} a_{\mu\nu}^{\lambda}f_N^{\lambda}. \tag{2}$$

An interesting feature of equation (1) is that $a_{\mu\nu}^{\lambda}=0$ unless $[\mu]$ of S_{N_1} is a sub-structure of $[\lambda]$ of S_N. This, in turn, permits us to consider the inverse problem of determining the possible $[\nu]$ of S_{N_2} whose product with a fixed $[\mu]$ of S_{N_1} yields a fixed $[\lambda]$ of S_N. It is possible to do this by starting with $[\lambda]$ of S_N and assigning the first N_1 of the particles to a fixed $[\mu]$. We then consider the skew diagram (Robinson 1961 p 48–51) $[\lambda]-[\mu]$. This skew diagram is decomposable into regular Young diagrams of S_{N_2} (Robinson 1961 p 64). The resulting Young diagrams can be obtained using rules similar to those for outer products:

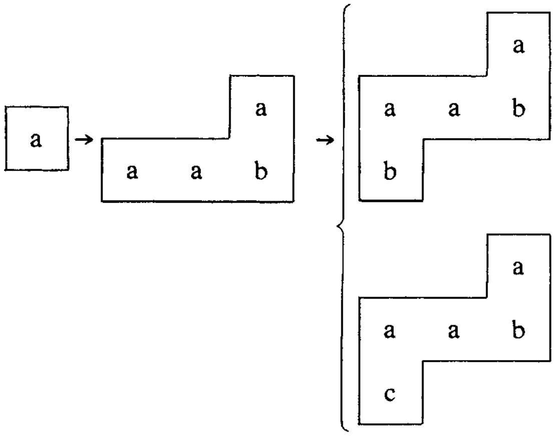
(i) Assign a set of indices a to the top row of the skew diagram $[\lambda]-[\mu]$.
(ii) Assign a, b to the row immediately below this such that (a) no two a's share a column, (b) at every stage the number of a's $\geqslant$ number of b's and (c) the resulting diagram is a lattice permutation,
(iii) For the third row, assign a, b, c so that the same conditions as in (ii) hold for all three indices. Proceed in this manner until all the rows of $[\lambda]-[\mu]$ are exhausted,

(iv) Obtain regular Young diagrams $[\nu]$ by assigning all a's of the skew diagram to the top row of $[\nu]$, all b's to the next row and so on.

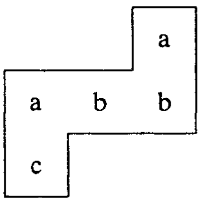
As an illustration, consider $[4^2, 2]-[3, 1^2]$ which has the skew shape



A self-explanatory diagramatic scheme of reduction satisfying the above rules is



In this process we observe that the structure



is not possible since it does not satisfy rule (iic) above. Using rule (iv) we then obtain

$$[4^2, 2]-[3, 1^2]=[3, 2]+[3, 1^2].$$

The correctness of the above reduction for the general case (Robinson 1961 p 64)

$$[\lambda]-[\mu]=\sum_k a^{\lambda}_{\mu\nu}(k)[\nu^{(k)}] \tag{3}$$

may be verified using the formula (Robinson 1961 p 49),

$$f^{\lambda-\mu}_{N_2}=N_2!\left|\frac{1}{(\lambda_i-i-\mu_j+j)!}\right|=\sum_k a^{\lambda}_{\mu\nu}(k)f^{\nu^{(k)}}_{N_2} \tag{4}$$

where λ_i and μ_i are row lengths of $[\lambda]$ and $[\mu]$ respectively and, in the determinant, $1/m!=1$ if $m=0$ and $1/m!=0$ if $m<0$. These considerations now permit us to define a subduction series as follows: Let $[\lambda]$ be an irrep of S_N which admits a regular substructure $[\mu]$ of S_{N_1} over the first N_1 particles. Then the set $\{[\nu^{(k)}] | [\nu^{(k)}]\in[\lambda]-[\mu]\}$

gives the possible irreps of S_{N_2} which determine the symmetries of the last N_2 particles. Equation (3) then yields the subduction sequence

$$[\nu^{(1)}], [\nu^{(2)}], [\nu^{(3)}], \ldots. \tag{5}$$

For notational convenience we assume that $[\nu^{(1)}] > [\nu^{(2)}] > \ldots$ are in alphabetical order by the order of decreasing row symmetry from top to bottom in the corresponding Young diagram (Robinson 1961 p 36).

The subduction series of equation (5) implies a corresponding linear transformation for the basis states spanning the respective irreps. Let $|[\lambda]; r\rangle$, $|[\mu]; s\rangle$, and $|[\nu^{(k)}]\tau_k; t\rangle$ be the basis spanning $[\lambda]$, $[\mu]$ and $[\nu^{(k)}]$ respectively with τ_k as an auxiliary index to distinguish between the multiply-occurring $[\nu^{(k)}]$. For a specific $[\nu^{(k)}]$ the subduction procedure implies the linear transformation (Harter and Patterson 1977, Sarma and Dinesha 1979)

$$[\lambda] \downarrow |[\mu]; s\rangle|[\nu^{(k)}]\tau_k; t\rangle = \sum_r S\begin{bmatrix} \lambda & \mu & \nu^{(k)} & \tau_k \\ r & s & t & \end{bmatrix} |[\lambda]; r\rangle \tag{6}$$

where the summation on the right is restricted to possible standard Young tableaux t_r^λ of $[\lambda]$ which admit a fixed subtableaux structure t_s^μ of $[\mu]$ over the first N_1 entries and

$$S\begin{bmatrix} \lambda & \mu & \nu^{(k)} & \tau_k \\ r & s & t & \end{bmatrix}$$

are the required subduction S coefficients. If an orthogonal basis is used for realising the irreps, the transformation of equation (6) can be chosen to be unitary so that we have

$$\sum_r S\begin{bmatrix} \lambda & \mu & \nu\tau \\ r & s & t \end{bmatrix} S\begin{bmatrix} \lambda & \mu & \nu\tau' \\ r & s' & t' \end{bmatrix} = \delta_{ss'}\delta_{tt'}\delta_{\tau\tau'} \tag{7}$$

$$\sum_{s,\tau,t} S\begin{bmatrix} \lambda' & \mu & \nu\tau \\ r' & s & t \end{bmatrix} S\begin{bmatrix} \lambda & \mu & \nu\tau \\ r & s & t \end{bmatrix} = \delta_{\lambda\lambda'}\delta_{rr'}, \tag{8}$$

by analogy with the corresponding relations for Clebsch–Gordan coefficients (Hamermesh 1962 p 270).

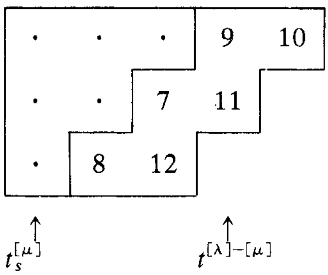
A straightforward scheme for determining the right-hand side of equation (6) is to apply the permutations P over the last N_2 entries to both sides of the equation. This leads to a system of linear equations in the unknown S coefficients weighted by the representation matrices for the permutations. Thus a knowledge of these representation matrices for the irreps $[\nu^{(k)}]$ and $[\lambda]$ should permit a determination of the required coefficients. This procedure, however, becomes quite complicated as N_2 becomes large since both the number and the size of the required representation matrices increase rapidly. An alternative is to use the maximal invariance subgroup of permutations of $|[\nu^{(k)}]\tau_k; t\rangle$ on both sides of equation (6). This would require only a knowledge of the representation matrices of $[\lambda]$ under the limited subset of permutations defined over the last N_2 particles. The difficulty with this approach is that now the linear combinations occurring on the right-hand side of equation (6) have contributions from all preceding tables spanning $[\nu^{(k)}]$ and also those from the higher irreps $[\nu^{(j)}]$ $(1 \leqslant j < k)$ occurring in the subduction series. Thus this procedure requires a complete knowledge of all these higher symmetry structures so that their contributions to the right-hand side of equation (6) for the given $[\nu^{(k)}]$ could be eliminated by orthogonalisation. Though this scheme also appears complicated it is worth noting that it is quite straightforward and does not involve all the permutations over the last N_2 particles. These aspects will now be considered in outlining the implementation of the procedure.

3. Calculation of the subduction coefficients

As a starting point for the determination of the S coefficients occurring in equation (6), consider first the product $[\lambda]\downarrow|[\mu];s\rangle\times|[\nu^{(k)}]\tau_k;1\rangle$ where the table $t_1^{[\nu^{(k)}]}$ is the first of the set in alphabetical ordering (Hamermesh 1962 p 201) spanning the irrep $[\nu^{(k)}]$. The Young basis corresponding to this table is invariant under each of the permutation groups of the first $\nu_1^{(k)}$, next $\nu_2^{(k)}$, etc particles where $\nu_1^{(k)}, \nu_2^{(k)}, \ldots, \nu_{p_k}^{(k)}$ is the partition of the last N_2 particles defining the irrep $[\nu^{(k)}]$. This invariance requires that the combinations $|[\lambda];r\rangle$ occurring on the right of equation (6) also reflect this symmetry. This, in turn, implies that in the skew portion of $[\lambda]$ we need only consider those tables in which no two of the first $\nu_1^{(k)}$, the second $\nu_2^{(k)}$ etc entries share a column. The number of ways in which such substructures of the skew portion of $[\lambda]$ can be partitioned defines the minimum number of irreps on the right-hand side of equation (5) which cannot be related by any permutation of this invariance subgroup of S_{N_2}. As an illustration of how such substructures occur, consider the restriction $[5,4,3]\downarrow[3,2,1]\otimes[3^2]$ of S_{12} to $S_6\otimes S_6$ where $[3^2]$ is an irrep occurring in the decomposition

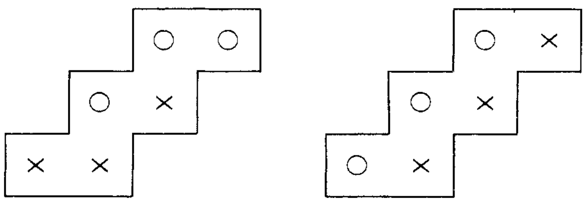
$$[5,4,3]-[3,2,1]=[4,2]+[4,1^2]+[3^2]+2[3,2,1]+[2^3].$$

Since every standard tableau $t_r^{[5,4,3]}$ spanning the above skew structure has a specified standard subtableaux structure $t_s^{[3,2,1]}$ we omit this portion of the former and indicate explicitly only the row locations of the last N_2 entries in the lattice permutation symbol. For example, in terms of Yamanouchi symbols (Hamermesh 1962 p 221)



$$\equiv|[\{s\}231123]\rangle^{[5,4,3]}.$$

Consider now the possible choices of substructure within the skew portion of $[5,4,3]$ which would admit symmetrisation over the first three and last three entries individually. An examination of the skew diagram readily yields the two possibilities



(I) (II)

where the entries from the set 7, 8, 9 are assigned to the positions marked $\bigcirc$ and those from the set 10, 11, 12 are assigned to the $\times$ positions so that the entries read increasing from left to right in each row and increasing from top to bottom in each column. It can be readily seen that the two structures above cannot be related by any permutation of 7, 8, 9 or 10, 11, 12 among themselves. Thus, assuming that we can determine the linear combinations of standard tableaux $t_r^{[5,4,3]}$ which have the required symmetry for each of the structures (I) and (II), we would still be left with two essential unknowns which define the linear combination of (I) and (II) required for the subduction. The explicit form of the linear combination may be represented as

$$[5,4,3]\downarrow|[3,2,1];s\rangle|[3^2];1\rangle=A|[\{s\}(112)(233)]\rangle^{[5,4,3]}+B|[\{s\}(123)(123)]\rangle^{[5,4,3]}$$

where a brace notation is introduced to indicate that each such symbol is a totally symmetric combination of basis states $|[\lambda];r\rangle$ obtained by permutations of the corresponding symbols within each bracket. An expansion of this symmetrised combination in terms of the individual Yamanouchi symbols follows on using the transformation properties of the Young orthogonal basis under elementary transpositions of S_N (cf Hamermesh 1962 p 221). As an outline of the procedure for obtaining the symmetrised linear combination consider the leading Yamanouchi symbol $[\{s\}11,\ldots,ij,\ldots]$ occurring in it. Using the correspondence between the Yamanouchi tableau and the standard Young tableau, we can readily work out the axial distance (cf definition of $(1/\rho)$, Robinson 1961 p 39) between the entries i and j in the Yamanouchi symbol as

$$d_{ij}=(\mu_i+n_i-i)-(\mu_j+n_j-j) \tag{8}$$

where μ_i and μ_j are partitions of μ and n_i and n_j are the number of i's and j's respectively to the left of the given pair i,j in the skew portion of the Yamanouchi symbol.

Consider now the linear combination

$$|[\{s\}11,\ldots,ij,\ldots]\rangle+C_{ij}|[\{s\}11,\ldots,ji,\ldots]\rangle \tag{9}$$

which by a suitable choice of the constant C_{ij} is to be made symmetric under the transposition of the consecutive entries corresponding to i and j of the Yamanouchi symbols occurring in the respective standard Young tableaux. Since the Young orthogonal representation matrices have a simple form for such elementary transpositions (Robinson 1961 p 38) and the required combination has to be symmetric, we obtain the result

$$1=\mp(1/d_{ij})+C_{ij}(d_{ij}^2-1)^{1/2}/d_{ij}, \tag{10}$$

which in turn fixes the value of C_{ij}:

$$C_{ij}=\left(\frac{d_{ij}\pm 1}{d_{ij}\mp 1}\right)^{1/2}. \tag{11}$$

In equations (10) and (11), the upper (lower) sign is to be used if $i<j (i\geqslant j)$. If $i=j$, the Yamanouchi symbol represents a Young tableau which is already symmetric under the interchange of the corresponding entries. If not, the expression (9) with C_{ij} as determined by equation (11) leads to a combination which is symmetric under this interchange. This procedure is repeated starting with each of the resulting tables until the complete symmetrised combination is obtained.

As an illustration of the procedure, consider the linear combination corresponding to $|[\{s\}(112)(233)]\rangle$ for the irrep $[5, 4, 3]$ of S_{12} where $\{s\}$ represents a Yamanouchi symbol for $[3, 2, 1]$ of S_6 defined over the first six particles. Omitting the label $[5, 4, 3]$ we have

$$\begin{aligned}|[\{s\}(112)(233)]\rangle = {} & |[\{s\}112233]\rangle + \sqrt{2}|[\{s\}112323]\rangle \\ & + \sqrt{6}|[\{s\}112332]\rangle + \sqrt{2}|[\{s\}121233]\rangle \\ & + 2|[\{s\}121323]\rangle + 2\sqrt{3}|[\{s\}121332]\rangle \\ & + 6|[\{s\}211233]\rangle + 2\sqrt{3}|[\{s\}211323]\rangle \\ & + 6|[\{s\}211332]\rangle\end{aligned}$$

where, for example, the numerical factor in front of $|[\{s\}112323]\rangle$ follows on using equation (8) since

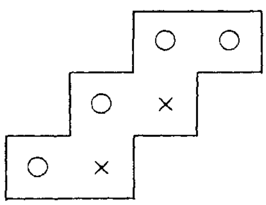
$$d_{23} = (2+1-2) - (1+0-3) = 3$$

so that on using equation (11)

$$\left(\frac{d_{23}+1}{d_{23}-1}\right)^{1/2} = (4/2)^{1/2} = \sqrt{2}.$$

Since every permutation $P \in S_N$ can be represented as a product of elementary transpositions, the invariance of the above combinations under the latter ensures a corresponding invariance under the former. As the above example illustrates, the final indeterminacy of the subduction coefficients for a restriction is related to the possible distinct ways of defining symmetrisable substructures in the skew portion of $[\lambda]$ corresponding to the partitions $\nu_1^{(k)}, \nu_2^{(k)}, \ldots, \nu_{p_k}^{(k)}$ of $[\nu^{(k)}]$. The reason for the occurrence of multiple symmetrisable substructures is, in turn, due to the fact that in this process we are essentially trying to obtain $[3^2]$ of S_6 from an outer product of the identity irreps of $S_3 \otimes S_3$. Such an outer product leads to the irreps $[6]$, $[5, 1]$, $[4, 2]$ and $[3^2]$ of $S_6 \supset S_3 \otimes S_3$ (cf equation (2.24) Robinson 1961 p 40). Since the subduction series for the example under consideration does not allow for the presence of the irreps $[6]$ and $[5, 1]$, the only possible contributions to the subduction coefficients for $[5, 4, 3] \downarrow |[3, 2, 1]; s\rangle|[3^2]; 1\rangle$ arise from the occurrence of $[4, 2]$ in the outer product $S_3 \otimes S_3$. Thus if we first determine the subduction coefficients for the higher symmetry product $[5, 4, 3] \downarrow |[3, 2, 1]; s\rangle|[4, 2]; 1\rangle$ and orthogonalise to the present result we can determine the unknown coefficient B in terms of A which can then be fixed using the normalisation condition.

To complete the example being considered let us obtain the subduction coefficients for $[5, 4, 3] \downarrow |[3, 2, 1]; s\rangle|[4, 2]; 1\rangle$. Using the same procedure as above, we first observe that there is only a single choice possible for the symmetrisable substructures corresponding to $|[4, 2];1\rangle$ occurring in $[5, 4, 3]-[3, 2, 1]$ as



This leads to

$$[5,4,3]\downarrow|[3,2,1];s\rangle|[4,2];1\rangle = N|[\{s\}(1123)(23)]\rangle$$
$$= N(|[\{s\}112323]\rangle + \sqrt{3}|[\{s\}113223]\rangle + \sqrt{2}|[\{s\}121323]\rangle$$
$$+\sqrt{3}|[\{s\}123123]\rangle + (3/\sqrt{2})|[\{s\}131223]\rangle + 3|[\{s\}132123]\rangle$$
$$+\sqrt{6}|[\{s\}211323]\rangle + 3|[\{s\}213123]\rangle + \sqrt{15}|[\{s\}231123]\rangle$$
$$+(15/2)^{1/2}|[\{s\}311223]\rangle + \sqrt{15}|[\{s\}312123]\rangle$$
$$+3\sqrt{5}|[\{s\}321123]\rangle + \sqrt{3}|[\{s\}112332]\rangle + 3|[\{s\}113232]\rangle$$
$$+\sqrt{6}|[\{s\}121332]\rangle + 3|[\{s\}123132]\rangle + (27/2)^{1/2}|[\{s\}131232]\rangle$$
$$+3\sqrt{3}|[\{s\}132132]\rangle + 3\sqrt{2}|[\{s\}211332]\rangle$$
$$+3\sqrt{3}|[\{s\}213132]\rangle + 3\sqrt{5}|[\{s\}231132]\rangle$$
$$+3(5/2)^{1/2}|[\{s\}311232]\rangle + 3\sqrt{5}|[\{s\}312132]\rangle$$
$$+3\sqrt{15}|[\{s\}321132]\rangle,$$

where $N = 1/4\sqrt{30}$ is a normalisation constant. Using this result, we can determine the unknown coefficient B occurring in the restriction $[5,4,3]\downarrow|[3,2,1];s\rangle|[3^2];1\rangle$ in terms of A as,

$$B = -3\sqrt{3}A/16\sqrt{2}$$

The final unknown A can be fixed by normalisation leading to the complete determination of the S coefficients for the required restriction.

The algorithm for obtaining the S coefficients for any restriction may now be stated as follows:

(i) We first obtain the reduction of $[\lambda]-[\mu]$ in terms of the irreps $[\nu^{(k)}]$ of S_{N_2}.

(ii) The subduction coefficients for the first Young basis of the highest symmetry irrep $[\nu^{(1)}]$ of S_{N_2} occurring in the above reduction are obtained using the symmetrisation technique. In this context it is worth noting that this irrep occurs once only in the reduction (Robinson 1961 p 40). This permits complete determination of the S coefficients for $[\lambda]\downarrow|[\mu];s\rangle|[\nu^{(1)}];1\rangle$.

(iii) If the S coefficients for any other table of this irrep are required they are obtained using the transformation properties of the Young basis under elementary transpositions of S_{N_2}.

(iv) The S coefficients of the next lower irrep $[\nu^{(2)}]$ are determined using the symmetrisation technique and orthogonalisation with respect to the necessary basis states occurring in the restriction $[\lambda]\downarrow[\mu]\otimes[\nu^{(1)}]$. Any ambiguity due to the multiple occurrence of $[\nu^{(2)}]$ in the reduction of $[\lambda]-[\mu]$ is resolved as was done by Hamermesh in his study of inner product reductions of S_N (cf example on pp 274–275, Hamermesh 1962).

(v) The procedure is continued until the required restriction is reached.

Since outer products are commutative, we can always choose $N_1 \geq N_2$ without any loss of generality and reduce the computational labour involved.

4. Discussion

The procedure outlined in § 3 is based essentially on a successive lowering scheme starting with the determination of the subduction coefficients for the first of the standard tables spanning the highest irrep occurring in the series for $[\lambda]-[\mu]$. This necessarily involves obtaining and storing a certain amount of information on the S coefficients for the irreps higher in row symmetry than the required one. In this sense the procedure is similar to that used by Moshinsky (1968) for obtaining the canonical basis spanning an irrep of the unitary group (cf Moshinsky 1968 pp 21–25).

The fact that the canonical basis spanning the irreps of the unitary group can be obtained using Wigner operators of S_N (Kaplan 1975 p 43, Sarma and Rettrup 1977, 1980, Sarma and Sahasrabudhe 1980, Paldus and Wormer 1979) should permit us to obtain the S coefficients for the restriction $U(n+m) \downarrow U(n) \otimes U(m)$ by using the results obtained from the procedure outlined in § 3. This aspect of the problem is currently under investigation.

Finally, it is to be noted that the logical structure of the present procedure is relatively simple and the storage requirements much less than those for direct or genealogical approaches. This should permit an easy computerisation of the scheme.

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